UK JUNIOR MATHEMATICAL CHALLENGE

THURSDAY 28th APRIL 2016

Organised by the **United Kingdom Mathematics Trust**
from the School of Mathematics, University of Leeds

http://www.ukmt.org.uk

Institute and Faculty of Actuaries

SOLUTIONS LEAFLET

This solutions leaflet for the JMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

For reasons of space, these solutions are necessarily brief. There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation:

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1. B The values of the expressions are: A 15, B 7, C 26, D −14, E $7\frac{1}{2}$. Of these, 7 is closest to 0.

2. A $60\,000 - 21 = 60\,000 - 20 - 1 = 59\,980 - 1 = 59\,979$.

3. E The number of laps is $5000 \div 400 = 50 \div 4 = 12\frac{1}{2}$.

4. C There are 41 years from January 1859 to January 1900 and a further 114 years to January 2014. So, since Åle died in August 2014, its age in years when it died was $41 + 114 = 155$.

5. A $\frac{1}{25} = \frac{4}{100} = 0.04$. So $\frac{1}{25} + 0.25 = 0.04 + 0.25 = 0.29$.

6. **C** Let there be g girls in Gill's school. Then there are $(g - 30)$ boys at the school. So $g + g - 30 = 600$. Therefore $2g = 630$, that is $g = 315$.

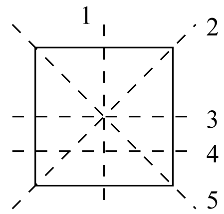
7. **A** As a distance of 8 km is roughly equal to 5 miles, 1.2 km $\approx \frac{1.2\times5}{8}$ miles $= \frac{6}{8}$ miles $= 0.75$ miles.

8. **A** By factorising the numerator, it is seen that;

$$\frac{2+4+6+8+10+12+14+16+18+20}{1+2+3+4+5+6+7+8+9+10} = \frac{2(1+2+3+4+5+6+7+8+9+10)}{1+2+3+4+5+6+7+8+9+10} = 2.$$

9. **E** All four numbers may be obtained: $36 = 20 + 16$; $195 = 201 - 6$; $207 = 201 + 6$; $320 = 20 \times 16$.

10. **D** When a square is folded exactly in half, the shape obtained is a rectangle or a right-angled isosceles triangle. So to determine which of the given shapes can be obtained from a second fold we need to test which shapes form a rectangle or a right-angled isosceles triangle when joined with the image formed when the shape is reflected about an edge. Of the options given, only D does not do this. Of the others, shape A is formed by using fold line 1 first, followed by fold line 3. For shape B the fold lines are 3 followed by 4. For shapes C and E, which are similar, the fold lines are 2 followed by 5.



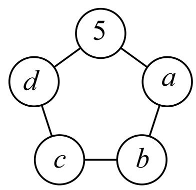
11. **C** A number is divisible by 4 if and only if its last two digits are divisible by 4. Since 34 is not divisible by 4, we deduce that 1234 is not a multiple of 4. Of the other options, 12 is even and so is a multiple of 2; the sum of the digits of 123 is 6, which is a multiple of 3, so 123 is a multiple of 3; 12 345 has a units digit of 5 and so is a multiple of 5. Finally, 123 456 is even and has a digit sum of 21, a multiple of 3. So 123 456 is a multiple of 2 and of 3 and is therefore a multiple of 6.

12. **B** Five hundred and twenty five thousand six hundred minutes is equal to

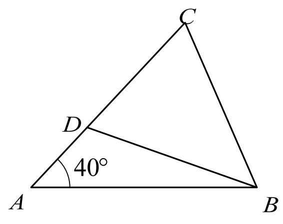
$$\frac{525\,600}{60}\text{ hours} = 8760\text{ hours} = \frac{8760}{24}\text{ days} = 365\text{ days}.$$

So the length of time in the song is the number of minutes in a year, unless it is a leap year.

13. **C** The position of the 5 is immaterial to the question asked, so let it be placed in the top circle. Now 4 differs by 1 from 5 so neither a nor d equals 4. Therefore either $b = 4$ or $c = 4$. It doesn't matter which it is, because the answer will be symmetric. So let $b = 4$. Since 3 differs by 1 from 4, neither a nor c can be 3, so $d = 3$. This leaves us with 1 and 2 to place. As 2 cannot be next to 3, $c \neq 2$ so $c = 1$ and $a = 2$. Therefore the sum of the numbers in the two circles adjacent to the circle containing 5 is $3 + 2 = 5$.

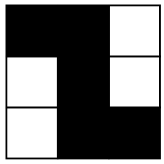


14. **D** As $AB = AC$, triangle ABC is isosceles. So $\angle ABC = \angle ACB = \frac{1}{2}(180° - 40°) = 70°$ as $\angle BAC = 40°$ and the angle sum of a triangle is 180°. Triangle BCD is also isosceles as $BD = BC$, so $\angle BDC = \angle BCD = 70°$. Considering triangle ABD: $\angle BDC = \angle DAB + \angle ABD$ as an exterior angle of a triangle is equal to the sum of the two interior opposite angles. So $\angle ABD = \angle BDC - \angle DAB = 70° - 40° = 30°$.



15. **E** All four expressions are perfect squares: $1^3 + 2^3 = 1 + 8 = 9 = 3^2$; $1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36 = 6^2$; $1^3 + 2^3 + 3^3 + 4^3 = 1 + 8 + 27 + 64 = 100 = 10^2$; $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 1 + 8 + 27 + 64 + 125 = 225 = 15^2$. (*It is not a coincidence that all four expressions are squares: the sum of the cubes of the first n integers is equal to the square of the sum of the first n integers for all positive integers n. For example*: $1^3 = 1^2$; $1^3 + 2^3 = (1 + 2)^2$; $1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2$; $1^3 + 2^3 + 3^3 + 4^3 = (1 + 2 + 3 + 4)^2$ *etc.*)

16. **B** We imagine all the squares being black and consider changing as few as possible to white in order to satisfy the conditions required. First note that the colour of the centre square has no effect on the symmetries involved. So we will leave that black. If we change one corner to white, the opposite corner must also be changed to white to give the rotational symmetry. The diagram still has reflective symmetry. If you instead try changing a non-corner square to white, the opposite one must be. And you again have reflective symmetry. That shows we need to change more than two squares. The rotational symmetry means that the next possiblity is to change 4 squares to white. And the diagram shown shows that it is possible, with four white squares, to have rotational but not reflective symmetry. That means that, in the problem as stated, the maximum number of black squares is 5.



17. **C** Initially there are 48 children of whom $\frac{3}{8}$ are boys and $\frac{5}{8}$ are girls, so there are 18 boys and 30 girls. When more boys join, there are still 30 girls but now they form $\frac{3}{8}$ of the total. So the total number of pupils is now $\frac{8}{3} \times 30 = 80$, of whom $80 - 30 = 50$ are boys. Hence the number of boys joining is $50 - 18 = 32$.

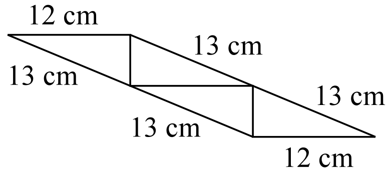
18. **D** First note that when two numbers are added together the only possible carry from any column is 1. Now, looking at the tens column of the sum, we see that $E + E$ leaves a total of E in the column. Since E is non-zero, the only way that this can happen is that there is a carry of 1 from the units column. So we have $1 + E + E = 10 + E$, so $1 + E = 10$, that is $E = 9$. Looking at the units column we see that $E + E = 18$, so $S = 8$ and there is a carry of 1 to the tens column. The addition sum may now be solved: $899 + 899 = 1798$. So $X = 7$.

19. **B** Let p be the total number of pears. Then $12 + \frac{p}{9} = \frac{1}{2}\left(p - \frac{p}{9}\right) = \frac{4p}{9}$. So $12 = \frac{3p}{9} = \frac{p}{3}$. Therefore $p = 3 \times 12 = 36$. So the number of pieces of fruit in each box is $\frac{12 + 36}{3} = 16$.

20. **E** The length of s, half the perimeter of the cyclic quadrilateral, is $\frac{1}{2}(4 + 5 + 7 + 10)$ cm $= 13$ cm. So the required area, in cm^2, is $\sqrt{(13-4)(13-5)(13-7)(13-10)} = \sqrt{9 \times 8 \times 6 \times 3} = \sqrt{9 \times 144} = 3 \times 12 = 36$.

21. **A** The area of the shaded triangle is $\frac{1}{2} \times 3 \times 6 = 9$. The area of the square grid is $6 \times 6 = 36$, and the area of the triangle which is not part of the area of the pentagon is $\frac{1}{2} \times 3 \times 3 = \frac{9}{2}$. So the area of the pentagon is $36 - \frac{9}{2} = \frac{63}{2}$. Hence the required fraction is $9 \div \frac{63}{2} = 9 \times \frac{2}{63} = \frac{2}{7}$.

22. **E** In order to join the four triangles together it is required to join together at least three pairs of edges, which consequently are not part of the perimeter of the resulting parallelogram. The four triangles have a total of 12 edges, so the maximum number of edges which can be part of the perimeter of the parallelogram is $12 - 3 \times 2 = 6$. For the perimeter to be as large as possible, all four 13 cm edges should be included together with two of the 12 cm edges, if this is possible. The diagram shows how it may be accomplished. So the largest possible perimeter is $(4 \times 13 + 2 \times 12)$ cm $= 76$ cm.

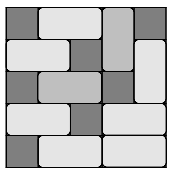


23. **B** Note first that the fourth square has side length 3, the fifth square has side length 4 and the sixth square has side length 7. As described in the question, the seventh square is placed alongside the sixth square, the fourth square and one of the first three unit squares. However, it may be seen that the side length of the seventh square is equal to the sum of the side lengths of the fifth and sixth squares, which is $4 + 7 = 11$. Similarly, the eighth square is placed along the fourth, fifth and seventh squares, but its side length is the sum of the side lengths of the sixth and seventh squares, which is $7 + 11 = 18$. The spiral sequence continues in the same way and therefore the side length of any subsequent square may be calculated by adding together the side lengths of the two previous squares in the sequence. So from the fourth square onwards the side lengths of the squares are 3, 4, 7, 11, 18, 29, 47, 76, 123, …. Hence the side length of the twelfth square is 123.

(*All of the positive integers in the sequence from the side length of the fourth square onwards are members of the sequence of Lucas numbers − a Fibonacci sequence with first term* 2 *and second term* 1.)

24. **D** First note that as there are four tiles to be placed and all three colours must be used, every arrangement of tiles consists of two of one colour and one each of the other two colours. Let the colours be R, G and B and consider the arrangements in which there are two tiles of colour R. These two tiles may be placed in six different ways: RR**, R*R*, R**R, *RR*, *R*R and **RR. For each of these arrangements of R tiles, there are two possible ways of placing the remaining G tile and B tile − the G tile may go in the first remaining space or the second remaining space and then there remains only one space for the B tile. So the number of arrangements in which there are two R tiles is $2 \times 6 = 12$. By the same reasoning, we see that there are 12 different arrangements in which there are two G tiles and 12 different arrangements in which there are two B tiles. So the total number of different arrangements is $3 \times 12 = 36$.

25. **D** First note that there are 25 squares on the board. As each domino occupies two squares, the number of squares left uncovered must be odd. The diagram on the right shows that it is possible for Beatrix to place the dominoes so that there are seven uncovered spaces when it is not possible for her to place any more dominoes.



Of the options given, it is not possible to obtain eight uncovered spaces as the number of them must be odd and it has been shown that seven uncovered spaces is possible so the correct answer is seven.

(*For a proof that it is not possible to obtain more than seven uncovered spaces, please see the extended solutions on the UKMT website*.)